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FINDING CONVEX SETS IN CONVEX POSITION

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Let $\mathcal{F}$ denote a family of pairwise disjoint convex sets in the plane. $\mathcal{F}$ is said to be in *convex position*, if none of its members is contained in the convex hull of the union of the others. For any fixed $k \geq 5$, we give a linear upper bound on $P_k(n)$, the maximum size of a family $\mathcal{F}$ with the property that any k members of $\mathcal{F}$ are in convex position, but no n are.

1. Introduction

In their classical paper [3], Erdős and Szekeres proved that any set of more than $\binom{2n-4}{n-2}$ points in general position in the plane contains n points which are in convex position, i.e., they form the vertex set of a convex n-gon. T. Bisztriczky and G. Fejes Tóth [1], [5] extended this result to families of convex sets.

Throughout this paper, by a *family* $\mathcal{F} = \{A_1, \dots, A_t\}$ we always mean a family of pairwise disjoint compact convex sets in the plane in *general position,* i.e., no three of them have a common supporting line. $\mathcal{F}$ is said to be in *convex position* if none of its members is contained in the convex hull of the union of the others, i.e., if $\operatorname{bd}\operatorname{conv}(\cup\mathcal{F})$, the boundary of the convex hull of the union of all members of $\mathcal{F}$, contains a point of the boundary of each A_i. Evidently, any two members of $\mathcal{F}$ are in convex position.

T. Bisztriczky and G. Fejes Tóth proved that there exists a function $P_3(n)$ such that if $|\mathcal{F}| > P_3(n)$ and any *three* members of $\mathcal{F}$ are in convex position,

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then $\mathcal{F}$ has n members in convex position. Improving their initial result, in [2] they showed that this statement is true with a function $P_3(n)$, triply exponential in n. This bound was recently improved to a simply exponential function by Pach and Tóth [7]. The best known lower bound for $P_3(n)$ is the classical lower bound for the Erdős-Szekeres theorem, $2^{n-2} \le P_3(n)$.

If any k members of $\mathcal{F}$ are in convex position, then we say that $\mathcal{F}$ satisfies *property* P_k. If no n members of $\mathcal{F}$ are in convex position, then we say that $\mathcal{F}$ satisfies *property* P^n. *Property* P_k^n means that both P_k and P^n are satisfied. Using these notions, the above cited result of Pach and Tóth states that if a family $\mathcal{F}$ satisfies property P_3^n, then $|\mathcal{F}| \le \binom{2n-4}{n-2}^2$.

T. Bisztriczky and G. Fejes Tóth [2] raised the following more general question. What is the maximum size $P_k(n)$ of a family $\mathcal{F}$ satisfying property P_k^n? Some of their bounds were later improved in [7]. The best known bounds are the following:

$$2^{n-2} \le P_3(n) \le \binom{2n-4}{n-2}^2 \qquad [4, 7]$$

$$2\left\lfloor \frac{n+1}{4} \right\rfloor^2 \le P_4(n) \le n^3 \qquad [7]$$

$$n-1+\left\lfloor \frac{n-1}{k-2} \right\rfloor \le P_k(n) \le n^2 \qquad \text{for } k \ge 5 \qquad [2]$$

$$n-1+\left\lfloor \frac{n-1}{k-2} \right\rfloor \le P_k(n) \le n \log n \qquad \text{for } k \ge 11 \qquad [2, 7]$$

In this note we give a linear upper bound on $P_k(n)$ for any $k \ge 5$.

Theorem. (i) *For any* $k \ge 6$ *we have*

$$P_k(n) \le n + \frac{1}{k-5} n,$$

and (ii)

$$P_5(n) \le 6n - 12.$$

2. Proof of Theorem

Let $\mathcal{F} = \{A_1, A_2, \dots, A_t\}$ be a family of pairwise disjoint convex sets in general position in the plane. Denote the convex hull of $\cup\mathcal{F} = \cup_{i=1}^{t} A_i$ by $\operatorname{conv}\mathcal{F}$. The boundary of $\operatorname{conv}\mathcal{F}$, $\operatorname{bd}\operatorname{conv}\mathcal{F}$, consists of finitely many boundary pieces of the A_i's, called *vertex-arcs*, connected by straight-line segments, called *edge-arcs*. (This terminology reflects the picture in the special case when every set A_i is a single point.)

The elements $A_i \in \mathcal{F}$ contributing at least one vertex-arc to the boundary of $\operatorname{conv}\mathcal{F}$ will be called *vertices of* $\operatorname{conv}\mathcal{F}$ or, simply, *vertices of* $\mathcal{F}$. If a vertex contributes to exactly one vertex-arc, then it is called a *regular vertex* of $\mathcal{F}$, otherwise it is an *irregular vertex*. If A is not a vertex, then it is said to be an *internal member* of $\mathcal{F}$.

Let A be an arbitrary vertex of $\mathcal{F}$ and $P \in \operatorname{bd}\operatorname{conv}\mathcal{F} \cap \operatorname{bd} A$. For any $A_i \in \mathcal{F}$ and $Q \notin A_i$ we say that Q is *above* A_i if A_i intersects the segment PQ. For any $A_i, A_j \in \mathcal{F}$ we say that A_j is *above* A_i if there is a $Q \in A_j$ such that Q is above A_i. Finally, A_j is said to be *strictly above* A_i if *any* $Q \in A_j$ is above A_i. We will refer to A and P as the *reference vertex* and *reference point* of $\mathcal{F}$.

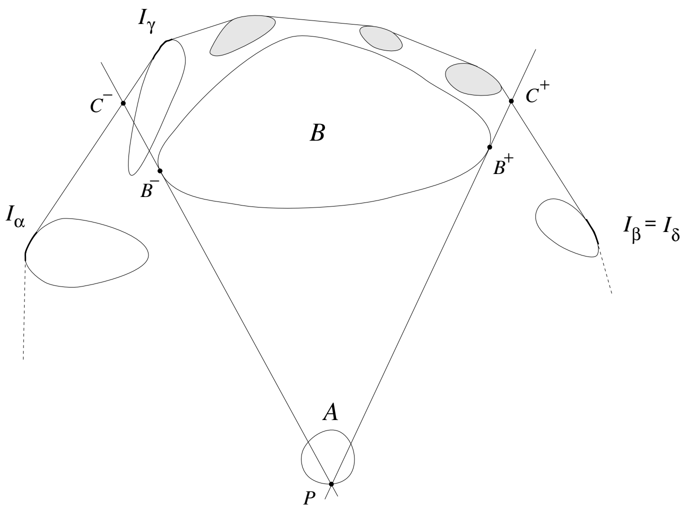

Fig. 1.

Lemma 1. *Let $\mathcal{F}$ be a family of pairwise disjoint convex sets in the plane satisfying property P_k, where $k \geq 3$. Then for any reference point P and any internal member B of $\mathcal{F}$, there are at least $k-3$ vertices of $\mathcal{F}$ strictly above B.*

Proof of Lemma 1. Let l^- and l^+ be the two tangent lines of B from P, with touching points B^- and B^+, respectively. Suppose that the triangle

PB^-B^+ is oriented clockwise, that is, l^- is the left, l^+ is the right tangent of B from P. Let C^- and C^+ be the intersections of $\operatorname{bd}\operatorname{conv}\mathcal{F}$ with l^- and l^+, respectively.

Let $I_1, I_2, \ldots, I_m$ be the vertex-arcs of $\operatorname{conv}\mathcal{F}$ in clockwise direction, such that $P \in I_1$ and let $A(I_1), A(I_2), \ldots, A(I_m)$ be the corresponding vertices. Note that some of $A(I_1), A(I_2), \ldots, A(I_m)$ may be identical as some vertices could have more than one vertex-arc. For any two points $x, y \in \operatorname{bd}\operatorname{conv}\mathcal{F}$, x *precedes* y and y *follows* x if and only if the clockwise order of P, x and y is Pxy on $\operatorname{bd}\operatorname{conv}\mathcal{F}$. Let

$$\alpha = \max\{i \mid I_i \text{ has a point which precedes } C^-\},$$
$$\beta = \min\{i \mid I_i \text{ has a point which follows } C^+\},$$
$$\gamma = \max\{i \mid i = \alpha \text{ or } i < \beta, A(I_i) \text{ intersects } PC^-\},$$
$$\delta = \min\{i \mid i = \beta \text{ or } i > \alpha, A(I_i) \text{ intersects } PC^+\}.$$

Since the sets are pairwise disjoint, $\gamma \le \delta$, so $\alpha \le \gamma \le \delta \le \beta$. Observe that

$$B \subset \operatorname{conv} A \cup \bigcup_{i=\gamma}^{\delta} A(I_i).$$

(see Fig. 1) Therefore, by property P_k, the collection $\mathcal{G} = \{A(I_i) \mid \gamma \le i \le \delta\}$ contains at least $k-1$ elements, Moreover, $\mathcal{G}' = \mathcal{G} \setminus \{A(I_\gamma), A(I_\delta)\}$ contains at least $k-3$ elements, all of them are vertices of $\mathcal{F}$, strictly above B. ∎

Lemma 2. *For any $k \ge 6$, $m > 0$, let $F_k(m)$ be the maximum number of elements of $\mathcal{F}$, a family of pairwise disjoint convex sets in the plane which satisfies property P_k, and has m vertices. Then*

$$F_k(m) \le \begin{cases} m & \text{if } 0 < m < 5 \\ m + \left\lfloor \frac{m-5}{k-5} \right\rfloor & \text{if } m \ge 5 \end{cases}$$

Proof of Lemma 2. For any fixed $k \ge 6$, we prove the statement by induction on m. If $\mathcal{F}$ has at most $k-1$ vertices, then by property P_k it does not have any internal member. This implies the statement for $m < k$. Suppose that the statement has already been proved for any $m' < m$ and that $\mathcal{F}$ has $m \ge k$ vertices. Let A and P be a reference vertex and reference point of $\mathcal{F}$. It is easy to see that there is an internal member B of $\mathcal{F}$ so that there is no other internal member above it (see [6]). By Lemma 1, there are at least $k-3$ vertices of $\mathcal{F}$, $\{A_1, A_2, \ldots, A_l\}$ strictly above B.

First suppose that one of them, say A_1, is an irregular vertex. Then A_1 separates a subfamily of $\{A_2,\dots,A_l\}$, from the rest of $\mathcal{F}$. Suppose without loss of generality that A_2 is in this subfamily. Deleting A_2 from $\mathcal{F}$, we do not create any new vertex so $\mathcal{F}\setminus\{A_2\}$ has one less members and one less vertices, and we are done by induction.

So we can suppose that all of $A_1,A_2,\dots,A_l$, $l\geq k-3$, are regular vertices, all of them are strictly above B and they appear in this clockwise order on $\operatorname{bd}\operatorname{conv}(\cup\mathcal{F})$. Let $\mathcal{F}'=\mathcal{F}\setminus\{B,A_2,\dots,A_{l-1}\}$ and let m' be the number of vertices of $\mathcal{F}'$. We deleted $l-2\geq k-5$ vertices of $\mathcal{F}$ and since there were no internal members of $\mathcal{F}$ above B, we did not get any new vertex. hence $m'\leq m-(k-5)$. Therefore, by the induction hypothesis,

$$|\mathcal{F}|\leq|\mathcal{F}'|+k-4\leq m'+\left\lfloor\frac{m'-5}{k-5}\right\rfloor+k-4\leq m+\left\lfloor\frac{m-5}{k-5}\right\rfloor. \qquad ■$$

Proof of Theorem (i) Let $\mathcal{F}$ be a family of pairwise disjoint convex sets in the plane satisfying property P_k^n, $6\leq k<n$. Observe that $\mathcal{F}$ has at most $n-1$ vertices, therefore, by Lemma 2, $|\mathcal{F}|\leq n-1+\left\lfloor\frac{n-6}{k-5}\right\rfloor<\frac{k-4}{k-5}n$. This concludes the proof of part (i). ■

Lemma 3. *Let $\mathcal{F}$ be a family of pairwise disjoint convex sets in the plane satisfying property P_5. If $\mathcal{F}$ has five vertices then it has at most one internal member.*

Proof of Lemma 3. Suppose first that A is an irregular vertex of $\mathcal{F}$. Then A divides $\mathcal{F}$ into two nonempty subfamilies, $\mathcal{F}',\mathcal{F}''\subset\mathcal{F}$ such that the members of $\mathcal{F}'$ are separated from the members of $\mathcal{F}''$ by A. Since both $\mathcal{F}'\cup\{A\}$ and $\mathcal{F}''\cup\{A\}$ has at most four vertices, by property P_5 they do not have internal members. Hence neither $\mathcal{F}$ has any internal member.

So let $A_1,A_2,\dots,A_5$ be the vertices of $\mathcal{F}$ with corresponding vertex-arcs $I_1,I_2,\dots,I_5$, in clockwise order. Suppose for contradiction that B and C are both internal members of $\mathcal{F}$. Let ℓ be a line which separates B and C. The line ℓ divides $\operatorname{bd}\operatorname{conv}(\cup\mathcal{F})$ into two parts, conv_B and conv_C, respectively. Since there are five vertex-arcs on $\operatorname{bd}\operatorname{conv}(\cup\mathcal{F})$, either conv_B or conv_C contains at most two of them, say, $I_2,I_3\subset\operatorname{conv}_B$. But then $B\subset\operatorname{conv}(A_1,A_2,A_3,A_4)$, contradicting property P_5. ■

Lemma 4. *For any $m>1$ let $\mathcal{F}$ be a family of pairwise disjoint convex sets in the plane which satisfies property P_5, and has m vertices. Then $|\mathcal{F}|\le 6m-6$.*

Proof of Lemma 4. We proceed by induction on m. The statement is trivial for $m\le 4$. Suppose that it has already been proved for any $m'<m$ and that $\mathcal{F}$ has m vertices.

Suppose first that A is an irregular vertex of $\mathcal{F}$. Then A divides $\mathcal{F}$ into two nonempty subfamilies, $\mathcal{F}',\mathcal{F}''\subset\mathcal{F}$ such that the members of $\mathcal{F}'$ are separated from the members of $\mathcal{F}''$ by A. Denote the number of vertices of $\mathcal{F}'\cup\{A\}$ and $\mathcal{F}''\cup\{A\}$ by m' and m'', respectively. Each vertex of $\mathcal{F}$ is either a vertex of $\mathcal{F}'\cup\{A\}$ or a vertex of $\mathcal{F}''\cup\{A\}$, except of A which is a vertex of both. Since there are no other vertices of $\mathcal{F}'\cup\{A\}$ and $\mathcal{F}''\cup\{A\}$, $m',m''<m$ and $m'+m''=m+1$. Apply the induction hypothesis for $\mathcal{F}'\cup\{A\}$ and $\mathcal{F}''\cup\{A\}$.

$$|\mathcal{F}|=|\mathcal{F}'\cup\{A\}|+|\mathcal{F}''\cup\{A\}|-1\le 6(m'+m'')-12=6m-6.$$

So we can assume that all vertices $A_1,A_2,\dots,A_m$ of $\mathcal{F}$ are regular vertices and $I_1,I_2,\dots,I_m$ are the corresponding vertex-arcs, in clockwise order. Substitute each A_i by $\mathrm{conv}(I_i)$. For simplicity we call the resulting family also $\mathcal{F}$. Clearly $\mathcal{F}$ sill has m vertices, and it is easy to see that property P_5 still holds.

We define a chain of families $\mathcal{F}\supset\mathcal{F}_1\supset\mathcal{F}_2\supset\cdots\supset\mathcal{F}_l$ such that $\mathcal{F}_l$ has no internal members. Throughout the process, $\mathcal{F}_i$ has m_i vertices, all regular vertices, and some consecutive quadruples of vertices $\{A_{i-1},A_i,A_{i+1},A_{i+2}\}$ may be marked. The number of marked quadruples will be denoted by k_i. At the beginning, $\mathcal{F}$ has m vertices and no marked quadruples, that is, $m_0=m$, $k_0=0$. Let A_1 be the reference vertex and P be the reference point of all $\mathcal{F}_j$.

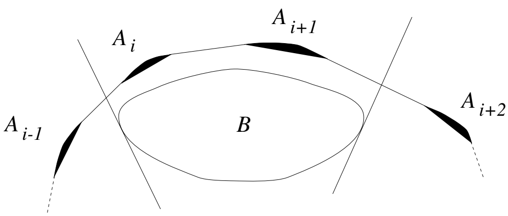

Fig. 2. Delete B and mark $\{A_{i-1},A_i,A_{i+1},A_{i+2}\}$

Inductive step. Let $A_1,A_2,\dots,A_{m_j}$ be the vertices of $\mathcal{F}_j$, in clockwise order. If $\mathcal{F}_j$ has no internal members, let $l=j$ and stop. Otherwise, let B be an internal member of $\mathcal{F}_j$ such that there is no other internal member above B. It follows from Lemma 1 that there are at least two consecutive vertices of $\mathcal{F}$, say A_j and A_{j+1}, strictly above B.

If the neighboring vertices, A_{i-1} and A_{i+2} are *not* strictly above B, then we say that B is *assigned* to the quadruple $\{A_{i-1}, A_i, A_{i+1}, A_{i+2}\}$. Then we *mark* $\{A_{i-1}, A_i, A_{i+1}, A_{i+2}\}$, delete B and repeat the inductive step (Fig. 2). Note that in this case $B \subset \mathrm{conv}(A, A_{i-1}, A_i, A_{i+1}, A_{i+2})$. Hence by Lemma 3, there was no set previously assigned to $\{A_{i-1}, A_i, A_{i+1}, A_{i+2}\}$. Therefore,

$$k_{j+1} = k_j + 1,\ m_{j+1} = m_j,\ |\mathcal{F}_{j+1}| = |\mathcal{F}_j| - 1. \tag{1}$$

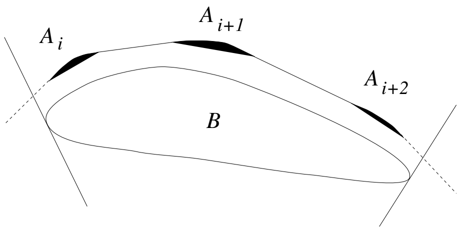

Fig. 3. Delete B and A_{i+1}

On the other hand, if there are at least three consecutive vertices, A_i, A_{i+1}, A_{i+2} strictly above B, then delete A_{i+1} and B (Fig. 3). We did not create any new vertex, so renumber the vertices $A_{i+2}, \ldots, A_{m_i}$ to $A_{i+1}, \ldots, A_{m_i-1}$ and repeat the inductive step. There were four quadruples which contained A_{i+1}, therefore the number of marked quadruples decreases by at most four. We did not create new vertices, so their number decreased by one. That is,

$$k_{j+1} \geq k_j - 4,\ m_{j+1} = m_j - 1,\ |\mathcal{F}_{j+1}| = |\mathcal{F}_j| - 2. \tag{2}$$

We claim that after each step $|\mathcal{F}_j| \leq 6m_j - k_j - 6$. Since there are m_j different consecutive quadruples and none of them can be marked twice, $k_j \leq m_j$. By property P_5, $m_l \geq 4$ therefore, $|\mathcal{F}_l| = m_l < 6m_l - k_l - 6$. Suppose that $|\mathcal{F}_{i+1}| \leq 6m_{i+1} - k_{i+1} - 6$. The connection between the parameters of $\mathcal{F}_j$ and $\mathcal{F}_{i+1}$ is described either by (
kUz1) or (2). In the case of (1),

$$|\mathcal{F}_j| = |\mathcal{F}_{i+1}| + 1 \leq 6m_{i+1} - k_{i+1} - 5 = 6m_j - (k_j + 1) - 5 = 6m_j - k_j - 6,$$

and in the case of (2),

$$|\mathcal{F}_j| = |\mathcal{F}_{i+1}| + 2 \leq 6m_{i+1} - k_{i+1} - 4 \leq 6(m_j - 1) - (k_j - 4) - 4 = 6m_j - k_j - 6.$$

This shows by induction that $|\mathcal{F}| \leq 6m - k - 6 = 6m - 6$. ∎

Proof of Theorem (ii). Let $\mathcal{F}$ be a family of pairwise disjoint convex sets in the plane satisfying property P_5^n. Since $\mathcal{F}$ has at most $n-1$ vertices, by Lemma 4, $|\mathcal{F}| \le 6(n-1)-6=6n-12$. This concludes the proof of part (ii). ∎

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